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The mass transfer and stability in systems with large concentration gradients—II. Hydrodynamic stability

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Abstract—The theoretical analysis about the influence of high concentration and large concentration gradients on the hydrodynamics stability of the flow in the laminar boundary layer has been done. The results obtained with the linear stability theory show that the stability of flows increases when density depends on concentration. The decrease of concentration gradient leads to decrease of stability. In the cases when the increase of concentration leads to increase (decrease) of viscosity, increase of stability can be observed, i.e. high concentrations lead to high (low) mass transfer rates in gases. Change in diffusivity does not influence stability. Additivity of the separated effects is observed. © 1997 Elsevier Science Ltd.

INTRODUCTION

It has been shown in the first paper [1] that the large concentration gradients in systems with intensive interphase mass transfer induce secondary flows on the phase boundary. The rate of these flows depends on concentration and its gradient:

$$v = -\frac{MD\rho^*}{\rho_0^*}\left[\frac{\partial}{\partial y}\left(\frac{c}{\rho}\right)\right]_{y=0}. \tag{1}$$

These secondary flows have an effect of 'injection' or 'suction' of fluid in the laminar boundary layer (depending on the direction of interphase mass transfer) and it leads to a decrease or increase of the hydrodynamic stability of the layer. The hydrodynamic stability has been examined in a number of investigations [2–5]. the critical Reynolds numbers have been determined in gas(liquid)–solid surface, gas–liquid, liquid–liquid and gas–liquid film flow systems. The results were obtained considering large concentration gradients and constant values of density, viscosity and diffusivity.

The analysis of equation (1) shows that a large concentration gradients 'injection' or 'suction' in or from the boundary layer depend not only on the concentration gradient, but on the concentration itself. This dependence manifests itself by means of concentration dependencies of density, viscosity and diffusivity.

The influence of the concentration and its gradient on the velocity distribution in laminar boundary layer has been determined in [1]. This gives an opportunity to research the hydrodynamic stability of the velocity profiles, which is the proposal of the present investigation. These investigations will be applied in the approximations of the linear stability theory for the almost parallel flows.

LINEAR ANALYSIS OF STABILITY

The flow in a laminar boundary layer will be examined as it was done in ref. [1]:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0$$

$$\rho\left(u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}\right)=\frac{\partial}{\partial y}\left(\rho D\frac{\partial c}{\partial y}\right)$$

$$x=0\quad u=u_0\quad c=c_0$$

$$y=0\quad u=0\quad v=-\frac{MD\rho^*}{\rho_0^*}\frac{\partial}{\partial y}\left(\frac{c}{\rho}\right)\quad c=c^*$$

$$y\to\infty\quad u=u_0\quad c=c_0. \tag{2}$$

The linear stability analysis considers a non-stationary flow (U, V, P), obtained as a combination of a basic stationary flow (u, v) and two-dimensional periodic disturbances (u_1, v_1, p_1) and with small amplitudes θ_1:

$$U(x,y,t)=u(x,y)+\theta_1 u_1(x,y,t)$$
$$V(x,y,t)=v(x,y)+\theta_1 v_1(x,y,t)$$
$$P(x,y,t)=\theta_1 p_1(x,y,t)$$
$$C(x,y,t)=c(x,y)+\theta_1 c_1(x,y,t). \tag{3}$$

The non-stationary flow (U, V, P), thus obtained, satisfies the full system of Navier–Stokes equations:

NOMENCLATURE

A	dimensionless wavenumber
C	dimensionless phase velocity
c	concentration [kg m^{-3}]
c_r	phase velocity, c_i increment factor
D	diffusivity [m^2 s^{-1}]
i	imaginary unit
Re	Reynolds number
u	velocity in x-direction [m s^{-1}]
v	velocity y-direction [m s^{-1}]
x	coordinate [m]
y	coordinate [m].

Greek symbols

α	wave number [m^{-1}]
β_i	increment factor
β_r	the circle frequency
λ	lengthwave [m]
μ	viscosity
ν	viscosity ($\nu = \mu/\rho$)
ρ	density.

$$\rho\frac{\partial U}{\partial t}+\rho U\frac{\partial U}{\partial x}+\rho V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right)$$

$$\rho\frac{\partial V}{\partial t}+\rho U\frac{\partial V}{\partial x}+\rho V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial V}{\partial y}\right)$$

$$\frac{\partial}{\partial x}(\rho U)+\frac{\partial}{\partial y}(\rho V) = 0 \tag{4}$$

with the following set of boundary conditions:

$$x=0 \quad U=u_0 \quad V=0 \quad P=p_0$$

$$y=0 \quad U=0 \quad V=-\theta_0\frac{\rho_0 D_0}{\Delta c_0}\frac{\partial}{\partial y}\left(\frac{c}{\rho}\right)$$

$$y\to\infty \quad U=u_0 \quad V=0 \quad P=p_0 \tag{5}$$

where

$$\theta_0 = \left(\frac{D\rho\Delta c_0}{\rho_0^*\rho_0 D_0}\right)_{y=0} \quad \Delta c_0 = c^*-c_0. \tag{6}$$

Linear approximations can be introduced into equations (4)–(6) for the dependencies of the density, viscosity and diffusivity from the concentration:

$$\rho = \rho_0(1-\bar{\rho}\bar{c}) \quad \mu = \mu_0(1-\bar{\mu}\bar{c}) \quad D = D_0(1+\bar{D}\bar{c}) \tag{7}$$

where

$$\bar{c} = \frac{c-\theta_1 c_1-c_0}{\Delta c_0}$$

while the parameters $\bar{\rho}$, $\bar{\mu}$ and $\bar{D}$ are small.

Consequential introduction of equations (2), (3) and (7) into equations (4) and (5) and after long transformations using the linear approximations for the small parameters θ_0, θ_1, $\bar{\rho}$, $\bar{\mu}$ and $\bar{D}$ leads to an equation describing the evolution of the superposed periodic flow (disturbance):

$$\frac{\partial u_1}{\partial t}+u\frac{\partial u_1}{\partial x}+v\frac{\partial u_1}{\partial y}+u_1\frac{\partial u}{\partial x}+v_1\frac{\partial u}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}+\nu_0\left(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}\right)$$

$$\frac{\partial v_1}{\partial t}+u\frac{\partial v_1}{\partial x}+v\frac{\partial v_1}{\partial y}+u_1\frac{\partial v}{\partial x}+v_1\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial y}+\nu_0\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}\right)$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y} = 0$$

$$x=0 \quad u_1=0 \quad v_1=0 \quad p_1=p_0$$

$$y=0 \quad u_1=0 \quad v_1=0$$

$$y\to\infty \quad u_1=0 \quad v_1=0 \quad p_1=p_0. \tag{8}$$

We have no equations for the concentration (c_1) in equation (8) because at the linear approximation for the small parameters θ_0 and θ_1 the disturbances do not influence the disturbances in the velocity (u_1, v_1).

The periodic disturbances can be considered as a running wave with a variable amplitude:

$$u_1 = G'(y)\exp i(\alpha x-\beta t)$$

$$v_1 = -i\alpha G(y)\exp i(\alpha x-\beta t) \tag{9}$$

where $G(y)$ is the amplitude of disturbance (regarding y); α and β/α are, respectively, its wavenumber and phase velocity:

$$\alpha = \frac{2\pi}{\lambda} \quad \beta = \beta_r+i\beta_i \quad \frac{\beta}{\alpha} = c_r-ic_i. \tag{10}$$

It is clearly seen from equations (9) and (10) that the amplitude of the disturbance decreases when $\beta_i < 0$ ($c_i < 0$), i.e. the basic flow is stable. At $\beta_i > 0$ ($c_i > 0$) the flow is unstable.

Hence, from equations (8) and (9) the equation of

Orr–Sommerfeld type is directly obtained (for almost parallel flow):

$$\left(u-\frac{\beta}{\alpha}\right)(G''-\alpha^2G)-\frac{\partial^2 u}{\partial y^2}G = -\frac{iv_0}{\alpha}(G'^{\,v}-2\alpha^2G''$$

$$+\alpha^4G)+\frac{i}{\alpha}\left[vG'''+\left(\frac{\partial^2 u}{\partial x\partial y}-\alpha^2 v\right)G'\right]$$

$$y=0 \quad G=0 \quad G'=0$$

$$y\to\infty \quad G=0 \quad G'=0. \tag{11}$$

STABILITY IN GASES

The analysis of stability needs introduction of velocity of the basic flow into equation (10). In ref. [1] it has been shown that in the case of gases:

$$u(x,y)=u_0\frac{\Phi'(\eta)}{\phi} \quad v=\frac{u_0\delta}{2x}\frac{\eta\Phi'(\eta)-\Phi(\eta)}{\phi}$$

$$\eta=\frac{y}{\delta} \quad \delta=\sqrt{\frac{D_0x}{u_0}} \quad \phi=1-\bar{\rho}F(\eta)$$

$$F(\eta)=\frac{c(x,y)-c_0}{c^*-c_0} \quad G(y)=\gamma(\eta) \quad \bar{\rho}\ll 1. \tag{12}$$

Introduction of equation (12) into (11) leads to

$$\left(\frac{\Phi'}{\phi}-C\right)(\gamma''-A_0^2\gamma)-\left(\frac{\Phi'''}{\phi}-2\bar{\rho}F'\frac{\Phi''}{\phi^2}-\bar{\rho}F''\frac{\Phi'}{\phi^2}\right)\gamma$$

$$=-\frac{i}{A_0Re_0}(\gamma'^{\,v}-2A_0^2\gamma''+A_0^4\gamma)+\frac{i}{2\varepsilon^2A_0Re_0}$$

$$\times\frac{\eta\Phi'-\Phi}{\phi}\gamma'''-\frac{i}{2\varepsilon^2A_0Re_0}\left(\frac{\eta\Phi'''+\Phi''}{\phi}\right.$$

$$\left.-\frac{2\bar{\rho}\eta F'\Phi''+\bar{\rho}\eta F''\Phi'+\bar{\rho}F'\Phi'}{\phi^2}+A_0^2\frac{\eta\Phi'-\Phi}{\phi}\right)\gamma' \tag{13}$$

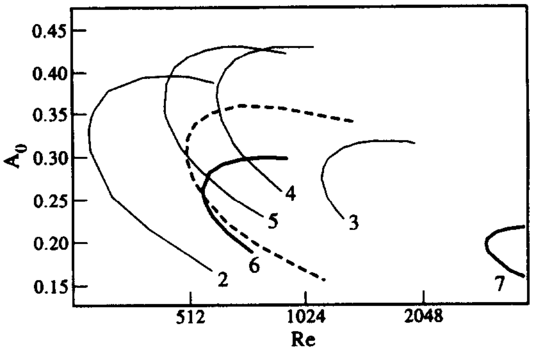
where

$$A=\alpha\delta \quad C=\frac{\beta}{\alpha u_0}=C_r+iC_i \quad Re_0=\frac{u_0\delta}{v_0}.$$

The solution of equation (12) has been accomplished as it was shown in ref. [2], where functions $\Phi(\eta)$ and $F(\eta)$ and their derivatives are taken from ref. [1].

The neutral curves of stability are plotted in Figs. 1–3. The critical Reynolds numbers Re_{cr}, corresponding wave velocities C_r, and wave numbers A are obtained. $C_{r\min}$ and $A_{\min}$ are also obtained from these results. We denote $C_{r\min}$ and $A_{\min}$ the minimal values for wave velocities and wave number at which the flow is stable at any Reynolds number Re, respectively. They are shown in Table 1 in dependence on the concentration of transferred substance ($\bar{\rho}$ and $\bar{\mu}$) and its gradient (θ), where $Re = 1.72\ Re_0$.

The results obtained show that the effect of the



(1) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0$;
(2) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0.2$;
(3) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = -0.2$;
(4) $\theta = 0, \bar{\rho} = 0.15, \bar{\mu} = 0$;
(5) $\theta = 0, \bar{\rho} = 0.15, \bar{\mu} = 0.2$;
(6) $\theta = 0, \bar{\rho} = -0.15, \bar{\mu} = 0$;
(7) $\theta = 0, \bar{\rho} = -0.15, \bar{\mu} = -0.2$

Fig. 1. The neutral curves of stability (Re_{cr}, A_0) for flows for gases in the laminar boundary layer under the conditions of high concentrations.

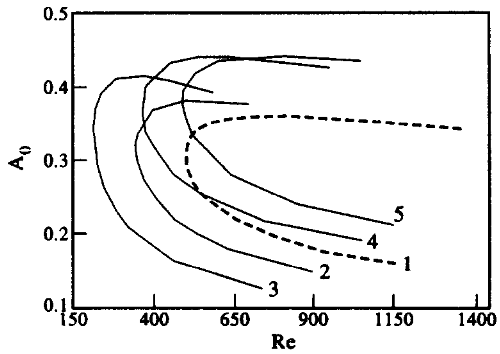
concentration dependencies of the viscosity ($\bar{\mu}$) is analogous to that one of the large concentration gradient (θ), while the change in the density ($\bar{\rho}$) has an insignificant effect and this dependence is not monotone.

STABILITY IN LIQUIDS

In the case of liquids the basic flow velocity is introduced into equation (10) as it has been defined in [1]:

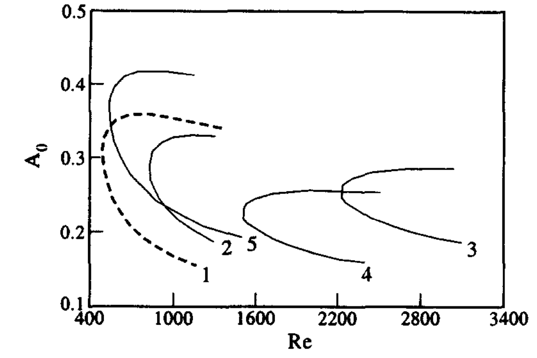
$$u(x,y)=u_0\frac{\Phi_1'(\eta_1)}{\phi} \quad v=\frac{u_0\delta_1}{2x}\frac{\eta_1\Phi_1'(\eta_1)-\Phi_1(\eta_1)}{\phi}$$

$$\eta_1=y\sqrt{\frac{u_0}{v_0x}} \quad \delta_1=\sqrt{\frac{v_0x}{u_0}}$$



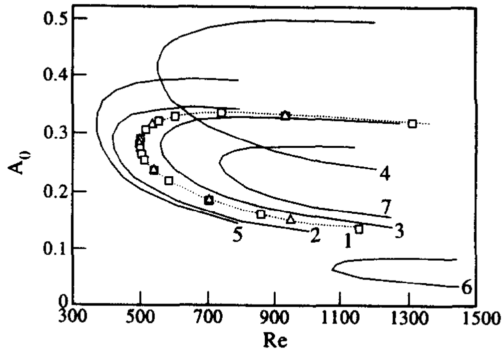
(1) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0$;
(2) $\theta = 0.3, \bar{\rho} = 0, \bar{\mu} = 0$;
(3) $\theta = 0.3, \bar{\rho} = 0, \bar{\mu} = 0.2$;
(4) $\theta = 0.3, \bar{\rho} = 0.15, \bar{\mu} = 0$;
(5) $\theta = 0.3, \bar{\rho} = 0.15, \bar{\mu} = 0.2$

Fig. 2. The neutral curves of stability (Re_{cr}, A_0) for flows of gases in the laminar boundary layer under the conditions of high concentrations and large concentration gradients.



(1) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0$;
(2) $\theta = -0.3, \bar{\rho} = 0, \bar{\mu} = 0$;
(3) $\theta = -0.3, \bar{\rho} = 0, \bar{\mu} = -0.2$;
(4) $\theta = -0.3, \bar{\rho} = -0.15, \bar{\mu} = 0$;
(5) $\theta = -0.3, \bar{\rho} = 0.15, \bar{\mu} = 0.2$

Fig. 3. The neutral curves of stability (Re_{cr}, A_0) for flows of gases in the laminar boundary layer under the conditions of high concentrations and large concentration gradients.



(1) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0, \bar{D} = 0$;
(2) $\theta = 0.3, \bar{\rho} = 0, \bar{\mu} = 0, \bar{D} = 0$;
(3) $\theta = -0.1, \bar{\rho} = 0, \bar{\mu} = 0, \bar{D} = 0$;
(4) $\theta = 0, \bar{\rho} = -0.15, \bar{\mu} = 0, \bar{D} = 0$
(5) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0.2, \bar{D} = 0$;
(6) $\theta = 0, \bar{\rho} = -0.15, \bar{\mu} = 0, \bar{D} = 0$
(7) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = -0.2, \bar{D} = 0$;
(□) $\theta = 0, \bar{\rho} = 0, \bar{\mu} = 0, \bar{D} = 0.3$
(△) $\theta = 0.3, \bar{\rho} = 0, \bar{\mu} = 0, \bar{D} = -0.3$

Fig. 4. The neutral curves of stability (Re_{cr}, A_0) for flows of liquids in the laminar boundary layer under the conditions of high concentrations.

$$\phi = 1 + \bar{\rho} F_1(\eta_1) \quad G(y) = \gamma_1(\eta_1) \tag{14}$$

where $\Phi_1(\eta_1)$ and $F_1(\eta_1)$ and their derivatives are obtained solving equation (2). Introduction of equation (14) into equation (11) leads to an equation of Orr–Sommerfeld type, which can be reached directly from equation (13) using the substitutions:

$$\Phi(\eta) = \Phi_1(\eta_1) \quad F(\eta) = F_1(\eta_1) \quad \gamma(\eta) = \gamma_1(\eta_1)$$

$$\eta = \eta_1 \quad A_0 = A_1 = \alpha\delta_1 \quad Re_0 = Re_1 = \frac{u_0 \delta_1}{\nu_0} \quad \varepsilon = 1. \tag{15}$$

The solution of this equation is done analogously to this one of equation (12), while $\Phi_1(\eta_1)$ and $F_1(\eta_1)$ and their derivatives have been obtained in [1]. The neutral curves of stability are shown in Fig. 4. The critical Reynolds numbers Re_{cr}, corresponding wave velocities C_r, and wave numbers A are obtained. $C_{r\min}$ and $A_{\min}$ are also obtained from these results. They are shown in Table 2 in dependence on the concentration of transferred substance ($\bar{\rho}, \bar{\mu}$ and $\bar{D}$) and its gradient (θ), where $Re = 1.72 Re_1$.

Table 1. Values of the critical Reynolds numbers Re_{cr}, corresponding wave velocities C_r, wavenumbers A and $C_{r\min}$, $A_{\min}$ obtained at the high concentrations (effects due to density ($\bar{\rho} \neq 0$) and viscosity ($\bar{\mu} \neq 0$)) and large concentration gradients ($\theta \neq 0$) in gases

No.	θ	$\bar{\rho}$	$\bar{\mu}$	Re_{cr}	$A_{\max}$	$C_{\max}$
1	0	0	0	501	0.356	0.407
2	0	0	0.2	285	0.394	0.445
3			−0.2	1135	0.315	0.352
4		0.15	0	608	0.429	0.356
5			0.2	443	0.428	0.373
6		−0.15	0	559	0.296	0.403
7			−0.2	2972	0.217	0.289
8	−0.3	0	0	1619	0.301	0.331
9			−0.2	2238	0.286	0.312
10		−0.15	0	1508	0.255	0.332
11		0.15	0.2	547	0.418	0.361
12	0.3	0	0	345	0.380	0.431
13			0.2	215	0.411	0.467
14		0.15	0	491	0.439	0.369
15			0.2	367	0.437	0.384

RESULTS AND DISCUSSIONS

The neutral curves of stability for flows of gases and liquids in the laminar boundary layer under the conditions of high concentrations and large concentration gradients are shown in Figs. 1–4. The scatter graphs in Fig. 4 having the same locus as the dotted line (1) are obtained in the case where the diffusivity depends on the concentration.

Data plotted in Figs. 1–4 allow us to determine the

Table 2. Values of the critical Reynolds numbers Re_{cr}, corresponding wave velocities C_r, wavenumbers A and $C_{r\min}$, $A_{\min}$ obtained at the high concentrations (effects due to density ($\bar{\rho} \neq 0$), viscosity ($\bar{\mu} \neq 0$) and diffusivity ($\bar{D} \neq 0$)) and large concentration gradients ($\theta \neq 0$) in liquids

No.	θ	$\bar{\rho}$	$\bar{\mu}$	$\bar{D}$	Re_{cr}	$A_{\max}$	$C_{\max}$
1	0	0	0	0	501	0.356	0.407
2	0.3	0	0	0	422	0.367	0.418
3	−0.1	0	0	0	564	0.351	0.398
4	0	0.15	0	0	556	0.518	0.358
5	0	−0.15	0	0	1073	0.102	0.392
6	0	0	0.2	0	373	0.416	0.414
7	0	0	−0.2	0	742	0.300	0.395
8(□)	0	0	0	0.3	502	0.357	0.406
9(△)	0	0	0	−0.3	501	0.357	0.406

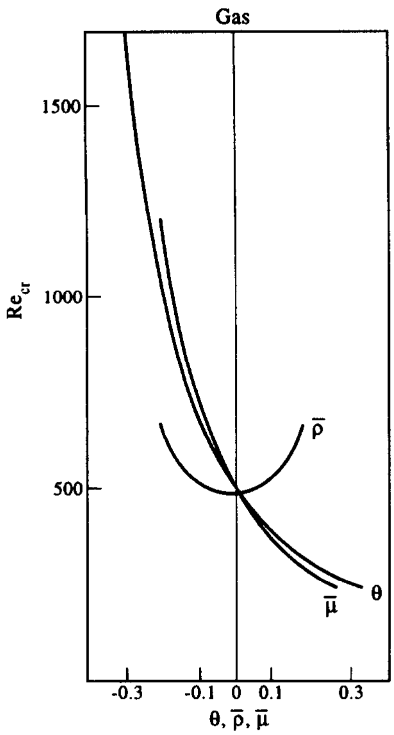


Fig. 5. The dependence of critical Reynolds numbers (Re_{cr}) from the high concentrations through the viscosity ($\bar{\mu}$) and density ($\bar{\rho}$), and the influence of the large concentration gradients (θ) in gases.

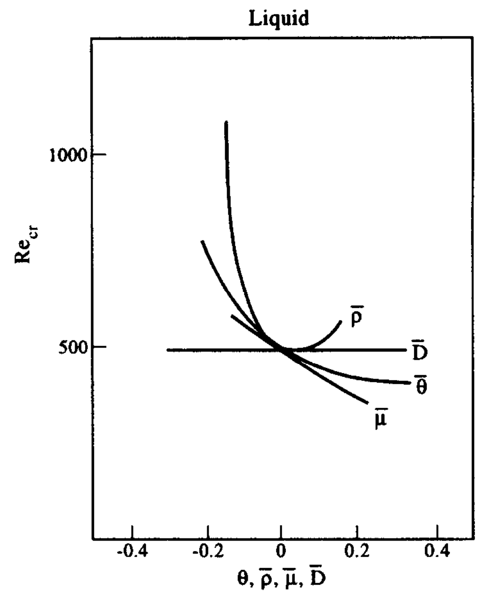


Fig. 6. The dependence of critical Reynolds numbers (Re_{cr}) from the high concentrations through the viscosity ($\bar{\mu}$) and density ($\bar{\rho}$) and diffusivity ($\bar{D}$), and the influence of the large concentration gradients (θ) in liquids.

critical Reynolds numbers (Re_{cr}), presented in Tables 1 and 2. They give us the opportunity to define (Figs. 5 and 6) the dependence of Re_{cr} from the parameters characterizing the concentration dependencies of density ($\bar{\rho}$), viscosity ($\bar{\mu}$), diffusivity ($\bar{D}$) and large concentration gradients (θ).

Data presented in Tables 1 and 2 and Figs. 5 and 6 show that in gases and liquids:

- the stability of flows (Re_{cr}) increases when density depends on concentration ($\bar{\rho} \neq 0$);
- decrease of concentration gradient (θ) leads to decrease of stability (Re_{cr});
- in the cases when the increase of concentration leads to increase (decrease) of viscosity, i.e. $\bar{\mu} > 0$ ($\bar{\mu} < 0$), we can observe increase of stability, i.e. high concentrations lead to high (low) mass transfer rates in gases;
- change in diffusivity ($\bar{D}$) does not influence stability;
- additivity of the separated effects is observed.

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